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Problems of Kuratowski Type

Note. This English translation of В. П. Солтан's paper, Задаче Типа Куратовского (in Russian), Mat. Issled., **65**(1982) 121–131, was prepared by Mark Bowron in September 2012. Some proofs have been modified.

While investigating the topological closure axioms, Kuratowski considered the following problem:

> *What is the largest number of distinct sets that can be obtained from an arbitrary subset in a topological space by repeatedly applying the closure and complement operations in any order?*

Kuratowski proved (see **[6]** and **[7,** 48–49]) that no more than 14 distinct sets can be obtained. In **[9]**, Zarycki showed that by repeatedly applying the complement and frontier operators to an arbitrary subset in a topological space, no more than six distinct sets can be obtained. Repeatedly applying the closure, complement and frontier operators produces no more than 34 distinct sets in a topological space **[3]**.

In this paper, the above results are extended to closure spaces. We also obtain some new results of the same type.

Recall that a mapping g is called a *closure operator* on a set X if for all subsets A, B of X we have

$$\begin{aligned} A \subset gA, &\quad (g \text{ is } \textit{extensive}) \\ ggA \subset gA, &\quad (g \text{ is } \textit{idempotent}) \\ A \subset B \Longrightarrow gA \subset gB. &\quad (g \text{ is } \textit{increasing}) \end{aligned}$$

A set X and closure operator g on X are together called a *closure space*. Let c denote the complement operator on X ($cA = X \setminus A$ for all $A \subset X$). Let i denote the interior operator $i = cgc$ on X. Let the frontier operator f be defined by the relation $fA = gA \cap gcA$.

Operator semigroups. We present diagrams representing the various operator semigroups generated by subsets of $\{g, c, i, f\}$. It is assumed that each of these semigroups contains the identity operator e ($eA = A$ for all $A \subset X$). The notation $\alpha \longrightarrow \beta$ ($\alpha \dashrightarrow \beta$) means that the element β of a semigroup is obtained by composing the element α with one of the operators g, c, i (the operator f). The symbols 0 and 1 represent the constant operators defined by the relations $0A = \varnothing$ and $1A = X$, respectively.

The semigroups generated by the singletons $\{g\}$, $\{c\}$, $\{i\}$, and $\{f\}$ each have a very simple form:

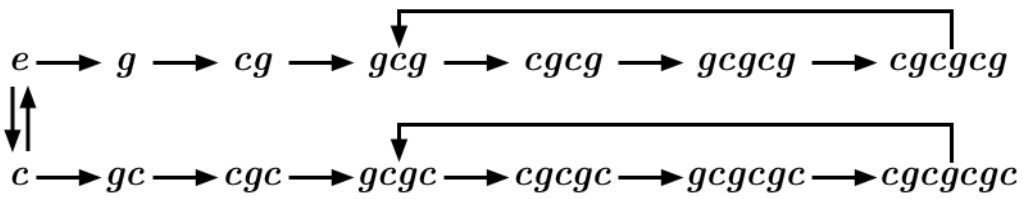
$$e \longrightarrow g, \quad e \rightleftarrows c, \quad e \longrightarrow i, \quad e \dashrightarrow f \dashrightarrow ff$$

The above diagrams follow from the basic identities $g^2 = g$, $c^2 = e$, $i^2 = i$, and the relation $fff = ff$ **[9]**.

Translator's note. Zarycki **[9]** assumed a topological closure. Some proofs generalize easily, but the proof given in **[9]** for $fff = ff$ does not. See **[4]** (Part 6 of Theorem C) instead.

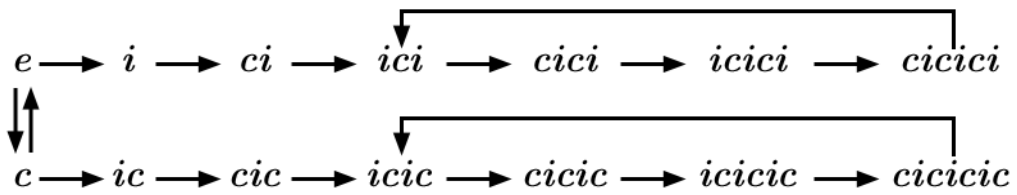
We now consider semigroups generated by any two of the operators g, c, i, f. Note, we shall omit arrows representing products of the form $c(c\alpha) = \alpha$ (except $c(ce) = e$) in what follows. Thus, while all of the diagrams display all of the semigroup elements, some do not display all of the semigroup products.

Semigroup generated by $\{g, c\}$. It follows from the known relation $gcgcgcg = gcg$ (see **[2,** 180] and **[4]**) that the semigroup generated by the operators g and c is represented by the diagram below.



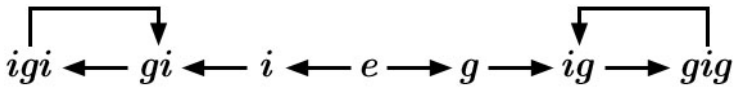
(1)

Semigroup generated by $\{i, c\}$. From diagram (1) and the relation $icicici = ici$, we get:



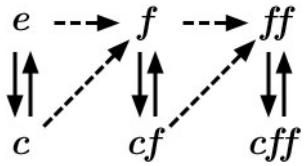
(2)

Semigroup generated by $\{g, i\}$. Similarly, from (1) we also get:



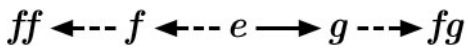
(3)

Semigroup generated by $\{f, c\}$. Noting the relations $fff = ff$ and $fc = f$ ($fcA = gcA \cap gccA = gcA \cap gA = fA$), we get the following diagram:



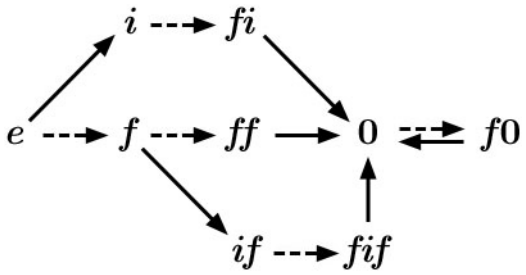
(4)

Semigroup generated by $\{g, f\}$. From the identities $fff = ff$, $ffg = fg$ [**9**], and $gf = f$ (the previous two identities hold by Theorem C, parts 5 and 2 in [**4**]), we deduce:



(5)

Semigroup generated by $\{i, f\}$. From the relations $ffi = fi$, $ifi = iff = 0$, $ff0 = g0 = f0$, and $if0 = 0$, we have:



(6)

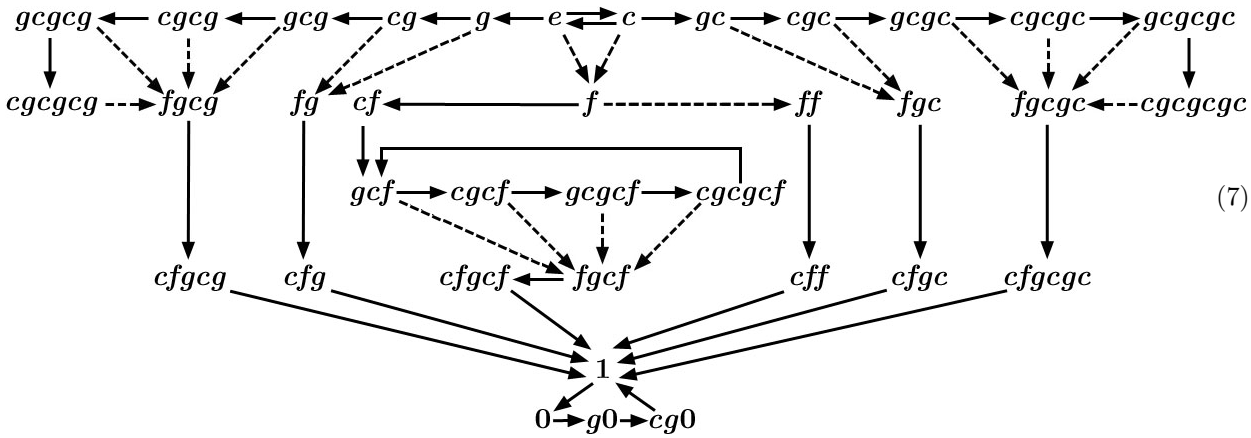
Indeed, $ffi = ffcgc = ffgc = fgc = fcgc = fi$. Since $gcfg = 1$ [**3**], we get $ifi = cgcfcgc = c(gcfg)c = 0$.

Translator's note. Though [**3**] assumes a topological closure, its proofs of the identities in the previous sentence all hold verbatim for our closure as well, with one exception: the third equals sign in $gcfgA = gc(gA \cap gcgA) = g(cgA \cup cgcgA) = gcgA \cup gcgcgA \supset gcgA \cup c(gcgA) = X$ must be replaced with the inclusion symbol "$\supset$". This clearly does not affect the argument.

Similarly $iff = 0$. The identities $ff0 = f0$, $if0 = 0$ follow from the relations $ff\varnothing = f(g\varnothing \cap gc\varnothing) = fg\varnothing = g\varnothing = f\varnothing$ and $ig\varnothing = c(gcg\varnothing) = cX = \varnothing$ (note that since $g\varnothing \subset gcg\varnothing$ and $g\varnothing \cup cg\varnothing = X$, it follows that $X \subset g(g\varnothing \cup cg\varnothing) \subset g(g\varnothing \cup gcg\varnothing) = gcg\varnothing$).

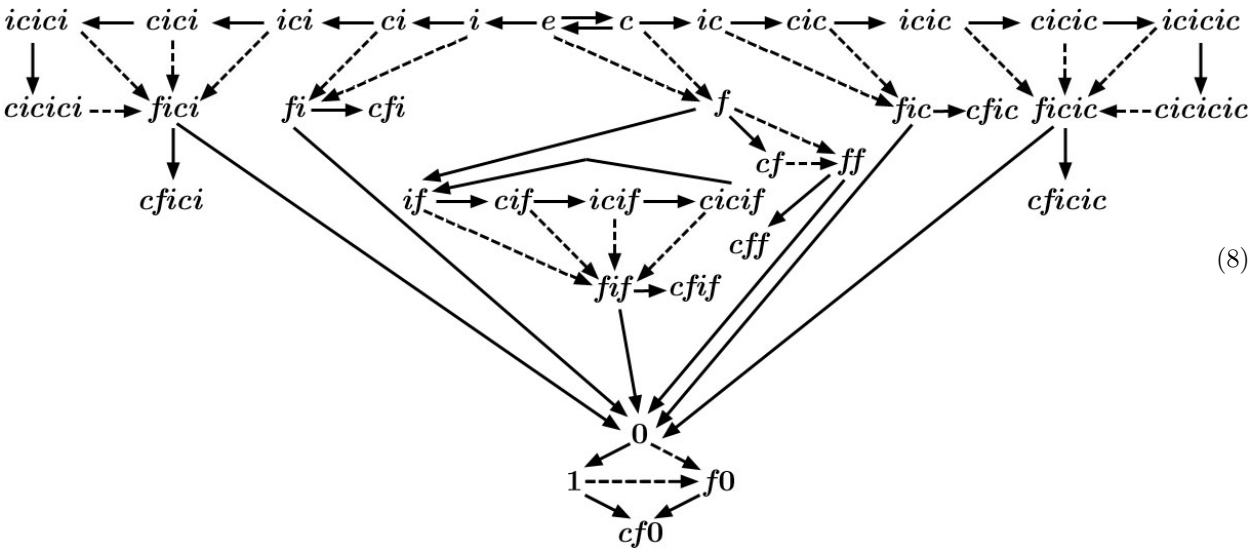
We now consider semigroups generated by any three of the operators g, c, i, f. Note, the semigroup generated by g, c, i clearly equals the (already displayed) one generated by g and c, since $i = cgc$.

Semigroup generated by $\{g, c, f\}$. From the identity $fgcgcg = fgcg$ **[3]** (this holds since $fgcgcgA = g(gcgcgA) \cap gc(gcgcgA) = gcgcgA \cap gcgA = gc(gcgA) \cap g(gcgA) = fgcgA$), we get:



(7)

Semigroup generated by $\{i, c, f\}$. From the above and the identities $icf = cgf = cf$, $icicif = if$, we get:

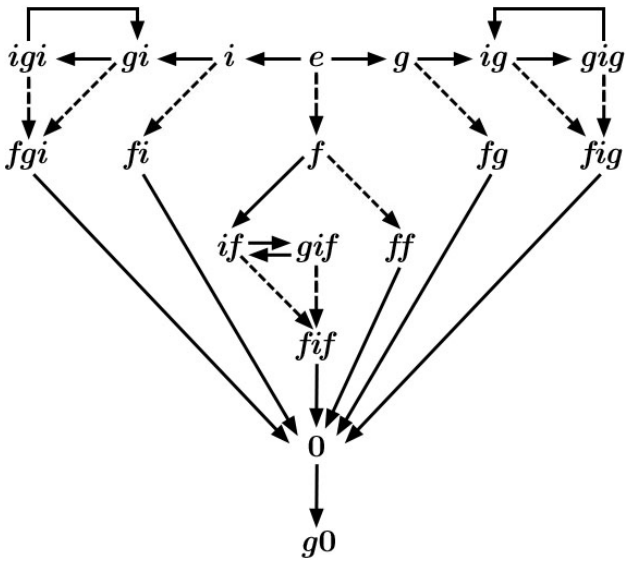


(8)

Semigroup generated by $\{g, i, f\}$. From the relations

$$ifg = c(gcfg) = c1 = 0, \quad igif = cgcgcgcf = cgcf = if,$$
$$fgig = fgcgcg = fgcg = fcgcg = fig, \quad figi = fcgcgcgc = fgcgc = fgi,$$

the semigroup generated by the operators g, i, f has the following form:



(9)

Numbers of generated sets. We now consider the various possible numbers of distinct sets generated by applying certain semigroups of operators to an arbitrary subset $A \subset X$. Specifically, we define:

semigroup	(1)	(3)	(4)	(5)	(6)	(7)	(9)
# distinct sets	$k(A)$	$l(A)$	$z(A)$	$p(A)$	$h(A)$	$m(A)$	$n(A)$

(Semigroups (1) and (2) are isomorphic; so are (7) and (8).) The semigroup diagrams imply the following inequalities:

$$\begin{aligned} &2 \le k(A) \le 14, \quad 1 \le l(A) \le 7, \qquad 2 \le z(A) \le 6, \qquad 1 \le p(A) \le 5, \\ &1 \le h(A) \le 9, \qquad 2 \le m(A) \le 36, \quad 1 \le n(A) \le 18. \end{aligned} \tag{10}$$

We claim these inequalities are sharp. The following modification of the example in **[3]** shows that the upper bounds cannot be reduced. Define g by $gA = \gamma A \cup \{7\}$ where γ is the closure operator for the usual topology on $\mathbb{R}$. (Note that g is not a topological closure operator, since $g\varnothing = \{7\}$.) Let

$$A = (0,1) \cup (1,2] \cup \{\mathbb{Q} \cap (2,3)\} \cup \{4\} \cup [5,6],$$

where $\mathbb{Q}$ is the set of rational numbers. It is easy to verify that in the closure space above, the set A realizes each of the upper bounds in (10). The lower bounds are realized by setting $A = \varnothing$ in any topological space. Hence the claim holds.

It is known **[8]** that $k(A)$ only takes even values from 2 to 14 and $l(A)$ can be any number from 1 to 7.

Theorem 1. For any subset $A \subset X$, the numbers $z(A)$ and $m(A)$ are even.

PROOF. Since $X \neq \varnothing$, no subset of X equals its own complement. Thus any finite family of subsets of X that is closed under c has even cardinality. Since c is in semigroups (4) and (7), Theorem 1 is proved.

An interesting problem is to determine which combinations of values are possible among $k(A)$, $l(A)$, $z(A)$, $p(A)$, $h(A)$, $m(A)$, and $n(A)$. The relationship between $k(A)$ and $l(A)$ has previously been studied. In the proof of Theorem 3 below, the following relationship will be established: $m(A) = 36 \iff n(A) = 18$.

To conclude this section, we give an example showing that the number of distinct sets obtained by repeatedly applying the closure, complement, and intersection operations can be infinite. Indeed suppose X is the interval $[-1,1]$ and the closure operator g is defined by $gA = A \cup \{x : -2x \in A\}$ for all $A \subset X$. Then for the set $A = [0,1]$, we get:

$$gA \cap cA = \left[-\tfrac{1}{2}, 0\right) := A_1, \quad gA_1 \cap cA_1 = \left[0, \tfrac{1}{4}\right] := A_2, \quad A_{2k} = \left[0, \tfrac{1}{2^{2k}}\right], \quad A_{2k+1} = \left[-\tfrac{1}{2^{2k+1}}, 0\right), \text{ for } k \geq 1.$$

Translator's note. The operator g defined above is not idempotent and hence not a closure operator. For example, $g(\{1\}) = \{-1/2, 1\}$ and $gg(\{1\}) = \{-1/2, 1/4, 1\}$. See Kuratowski **[6,** 197] for an example of a subset in a topological space that generates infinitely many distinct sets under closure, complement, and intersection.

Finite closure spaces. We now investigate the cardinalities of both X and the family of closed subsets $\mathcal{F} = \{A \subset X : gA = A\}$ when the maximum value of $k(A)$, $l(A)$, $z(A)$, $p(A)$, $h(A)$, $m(A)$, $n(A)$ is attained for some $A \subset X$. The theorem below addresses the possible values of $|X|$.

Theorem 2. If the closure space X contains a set A such that

$$\begin{aligned}
&1) \quad k(A) = 14, \quad \text{then } |X| \geq 6,\\
&2) \quad l(A) = 7, \quad \text{then } |X| \geq 6,\\
&3) \quad z(A) = 6, \quad \text{then } |X| \geq 3,\\
&4) \quad p(A) = 5, \quad \text{then } |X| \geq 4,\\
&5) \quad h(A) = 9, \quad \text{then } |X| \geq 7,\\
&6) \quad m(A) = 36, \quad \text{then } |X| \geq 9,\\
&7) \quad n(A) = 18, \quad \text{then } |X| \geq 9.
\end{aligned}$$

Proof. Parts 1 and 2 are proved in **[1]**. Part 3 is trivial since $|X| < 3 \Longrightarrow |2^X| \leq 4$.

Translator's note. Neither part 1 nor part 2 is proved in **[1]** for a general closure operator. See the appendix for a proof of part 1. Part 2 follows since it is equivalent to part 1.

We now prove part 4. We again have $|X| \geq 3$. Suppose $|X| = 3$. Since $fgA \subset fA \subset gA$ and $ffA \subset fA$, it follows that $gA = X$ and $|fA| = 2$. Let $X = \{a, b, c\}$ and $fA = \{b, c\}$. If $|A| = 1$, then $A = \{a\}$ since $iA \cup fA = gA = X$, and $ffA = fA \cap gcfA = \{b, c\} = fA$, which cannot hold. Without loss of generality, the only remaining possibility is $A = \{a, b\}$. Then $fgA = g\varnothing$ and $ffA = g(\{a\}) \cap \{b, c\}$. If $g(\{a\}) = \{a\}$, then $g\varnothing = \varnothing$ (since $ffA = \varnothing \Longrightarrow g\varnothing = gffA = ffA = \varnothing$), hence $fgA = ffA$. Thus $|g(\{a\})| > 1$. If $g(\{a\}) = X$, then $ffA = fA$. Thus $|g(\{a\})| = 2$. Hence $g(\{a\}) = \{a, c\}$, since $g(\{a\}) = \{a, b\} \Longrightarrow gA = A$. Thus $g(\{c\}) = \{c\}$ since both $g(\{a\}) = \{a, c\}$ and $fA = \{b, c\}$ imply that $g(\{c\}) \subset \{a, c\} \cap \{b, c\} = \{c\}$. But this implies $fA = \{c\}$, contradicting $fA = \{b, c\}$. Conclude $|X| \geq 4$.

We now prove part 5. Note that $fifA = gifA \cap gcfA \subset gfA \cap gcfA = ffA$. Since by diagram (6)

$$\varnothing \underset{\neq}{\subseteq} g\varnothing \underset{\neq}{\subseteq} fifA \underset{\neq}{\subseteq} ffA,$$

we get $|ffA| \geq 3$. Claim $|ifA| \geq 2$. Suppose $ifA = \{a\}$. Then $gcfA = X \setminus \{a\}$. Exactly one of the sets A, cA does not contain the point a and is therefore contained in the closed set $X \setminus \{a\}$. But this is impossible since $\{a\} = ifA \subset fA = gA \cap gcA$. Hence the claim holds. Since $ffA \cup ifA \subset fA$ and $ffA \cap ifA = \varnothing$, it follows that $|fA| \geq |ffA| + |ifA| \geq 5$. If $iA \cup fA = X$, then $fiA = g(iA) \cap g(ciA) = gcfA \cap gfA = ffA$. Therefore, since $iA \cap fA = \varnothing$, we get $|X| \geq |fA| + |iA| + 1 \geq 7$.

We now prove part 7. We begin by showing that $fiA \subset ffA$. We have $fiA = gcgcA \cap gcA \subset gA \cap gcA = fA$. Since $iA \cap fA = \varnothing$, we get $iA \subset cfA$ and hence $fiA \subset giA \subset gcfA$. Therefore, $fiA \subset gfA \cap gcfA = ffA$. We now show that $|fiA| \geq 3$. Suppose $|fiA| = 2$, $fiA = \{x, y\}$, $g\varnothing = \{x\}$. Then $giA = iA \cup fiA = iA \cup \{x, y\}$ and $igiA = cgc(iA \cup \{x, y\})$. By assumption we have $igiA \neq iA$. Hence $y \notin gc(iA \cup \{x, y\})$ since $iA \underset{\neq}{\subseteq} igiA \underset{\neq}{\subseteq} iA \cup \{x, y\}$ and $\{x\} = g\varnothing \subset gc(iA \cup \{x, y\})$. But then $fgiA = \{x\} = g\varnothing$, contradicting our assumption. Therefore $|fiA| \geq 3$ and $|ffA| \geq |fiA| + 1 \geq 4$. Hence $|fA| \geq |ffA| + |ifA| \geq 6$.

We show that in fact $|fA| > 6$. Suppose $|fA| = 6$. Then $|ffA| = 4$ and $|ifA| = 2$. Note, $fA = ffA \cup ifA$ and $fifA \subset ffA$.

If $fiA \cup fifA = ffA$, then

$$\begin{aligned} gA &= iA \cup fA \\ &= iA \cup (fiA \cup fifA \cup ifA) \\ &= (iA \cup fiA) \cup (fifA \cup ifA) \\ &= g(iA) \cup g(ifA) \\ &\subset g(iA \cup ifA) \\ &\subset gi(iA \cup fA) \\ &= gigA, \end{aligned}$$

which is impossible. Therefore, $fiA \cup fifA \subsetneqq ffA$.

From this we get $fifA \subset fiA$ since we have both $|fiA \cup fifA| < |ffA| = 4$ and $|fiA| \geq 3$. Thus we can assume $g\varnothing = \{x\}$, $fifA = \{x, y\}$, $fiA = \{x, y, z\}$, and $ffA = \{x, y, z, v\}$. Since

(a) $giA = iA \cup fiA = iA \cup \{x, y, z\}$,

(b) $\{x, y\} = fifA \subset gifA = gi(gA \cap gcA) \subset gigA \cap gig(cA) \subset gig(cA) = c(igiA)$, and

(c) $iA \subsetneqq igiA \subsetneqq iA \cup \{x, y, z\}$,

it follows that $igiA = iA \cup \{z\}$. But then $fgiA = giA \cap gcgiA = giA \cap c(igiA) = \{x, y\} = fifA$, which is impossible. This contradiction implies $|fA| > 6$. Therefore, $|X| \geq |fA| + 2 \geq 9$.

If $m(A) = 36$, then $n(A) = 18$. Hence $|X| \geq 9$. This completes the proof of Theorem 2.

The next theorem addresses the possible values of $|\mathcal{F}|$ when various maximal families exist in X.

Theorem 3. If the closure space X contains a set A such that

$$\begin{array}{lrll} 1) & k(A) &= 14, & \text{then } |\mathcal{F}| \geq 14, \\ 2) & l(A) &= 7, & \text{then } |\mathcal{F}| \geq 14, \\ 3) & z(A) &= 6, & \text{then } |\mathcal{F}| \geq 2, \\ 4) & p(A) &= 5, & \text{then } |\mathcal{F}| \geq 5, \\ 5) & h(A) &= 9, & \text{then } |\mathcal{F}| \geq 13, \\ 6) & m(A) &= 36, & \text{then } |\mathcal{F}| \geq 24, \\ 7) & n(A) &= 18, & \text{then } |\mathcal{F}| \geq 24. \end{array}$$

Translator's note. Inequalities 6) and 7) were incorrectly stated as $|\mathcal{F}| \geq 25$ in the original.

Proof. We first prove part 2. The condition $l(A) = 7$ implies that the sets $cgcA$, $cgcgcgcA$, $gcgcA$, $cgcgA$, $gcgcgA$, and gA are all distinct, with $gA \neq X$ and $cgcA \neq \varnothing$ (these inequations hold by Theorem 1 parts 5 and 6 in the author's 1980 paper, *On Kuratowski's Problem*). It follows that the sets

$$X,\ gA,\ gcA,\ gcgA,\ gcgcA,\ gcgcgA,\ gcgcgcA \tag{11}$$

are distinct. We claim that the sets

$$\left.\begin{array}{c} gA \cap gcA,\ gA \cap gcgA,\ gA \cap gcgcgcA,\ gcA \cap gcgcA, \\ gcA \cap gcgcgA,\ gcgA \cap gcgcA,\ gcgcgA \cap gcgcgcA \end{array}\right\} \tag{12}$$

are new. Any inclusion of the form $\alpha A \subset \beta A$ where α is an odd Kuratowski operator (has an odd number of c's) and β is an even Kuratowski operator (has an even number of c's) — or vice-versa — implies the existence of a chain $cU \subset \alpha A \subset \beta A \subset U$ where $U = gA$ or $U = gcA$. Since this implies $gA = X$ or $gcA = X$, no such inclusion holds when $l(A) = 7$. For example, $gcgA \subset gcgcA \Longrightarrow c(gA) = i(cA) \subset gi(cA) = gcgA \subset gcgcA = giA \subset gA \Longrightarrow gA = X$.

Note, if a set from (12) equals a set from (11), then applying gc to both sides of the equation yields an inclusion $\alpha A \subset \beta A$ of the type described above. For example, if $gA \cap gcA = gcgcgA$, then $gcgA \subset gcgA \cup gcgcA \subset g(cgA \cup cgcA) = gc(gA \cap gcA) = gcgcgcgA = gcgcA$. Therefore no set in (12) equals any set in (11).

See the appendix for a proof that all sets in (12) are distinct. Since all of the sets in (11) and (12) are distinct — and closed — we conclude $|\mathcal{F}| \geq 14$.

If $k(A) = 14$, then $l(A) = 7$. Hence $|\mathcal{F}| \geq 14$.

Translator's note. The implication $n(A) = 18 \Longrightarrow m(A) = 36$ holds in essentially the same way that $l(A) = 7 \Longrightarrow k(A) = 14$ does. Suppose $n(A) = 18$. Let $\mathcal{E}$ denote the ten nonempty sets generated by the operators in the intersection of the blue and red sections in diagram (16) (see the appendix). Since these are distinct, so are their complements. Moreover since each set in $\mathcal{E}$ is contained in fA, if any of the complements is contained in any set in $\mathcal{E}$, we get a chain $cfA \subset \alpha A \subset \beta A \subset fA$, contradicting $fA \neq X$. The equation $m(A) = 36$ follows. Thus $n(A) = 18 \Longleftrightarrow m(A) = 36$.

If $z(A) = 6$, then the sets $fA, ffA \in \mathcal{F}$ are distinct. Hence $|\mathcal{F}| \geq 2$.

We now prove part 4. If $p(A) = 5$, then the sets gA, fgA, fA, $ffA \in \mathcal{F}$ are distinct. If $gA \neq X$, then $ffA \subset fA \subset gA$ and $fgA \subset gA$ imply X is new. If $gA = X$, then $gcA \neq X$ (otherwise $fA = gA$). Hence, the sets $gA = X$, $fgA = g\varnothing$, $fA = gcA$, $ffA = gcA \cap gcgcA$, and $gcgcA$ are all distinct. Therefore $|\mathcal{F}| \geq 5$.

See the appendix for proofs of parts 5 and 7.

Translator's note. The proofs of Theorem 3 parts 5 and 7 in the original are both incomplete.

If $m(A) = 36$, then $n(A) = 18$. Hence $|\mathcal{F}| \geq 24$.

This completes the proof of Theorem 3.

The following examples show that the inequalities in Theorems 2 and 3 are sharp.

1. Let $X = \{a, b, c, d, e, f\}$,

$$\begin{aligned}
\mathcal{F} = \{&\{a, b, c, d, e, f\}, && 1\\
&\{a, b, c, d, e\ \}, && 2\\
&\{\ b, c, d, e, f\}, && 3\\
&\{a, b, c, d\ \}, && 4\\
&\{\ b, c, d, e\ \}, && 5\\
&\{\ \ c, d, e, f\}, && 6\\
&\{a, b,\ \ d\ \ \}, && 7\\
&\{\ b, c, d\ \ \}, && 8\\
&\{\ \ c, d, e\ \}, && 9\\
&\{\ \ \ d, e, f\}, && 10\\
&\{\ b,\ \ d\ \ \}, && 11\\
&\{\ \ c, d\ \ \}, && 12\\
&\{\ \ \ d, e\ \}, && 13\\
&\{\ \ \ d\ \ \}\}, && 14
\end{aligned}$$

and $A = \{a, c, e\}$. Then $|X| = 6$, $|\mathcal{F}| = 14$, and $k(A) = 2l(A) = 14$.

2. Let $X = \{a, b, c\}$, $\mathcal{F} = \{X, \{a\}\}$, and $A = \{a, b\}$. Then $|X| = 3$, $|\mathcal{F}| = 2$, and $z(A) = 6$.

3. Let $X = \{a, b, c, d\}$, $\mathcal{F} = \{X, \varnothing, \{b\}, \{a, b\}, \{b, c, d\}\}$, $A = \{a, c\}$. Then $|X| = 4$, $|\mathcal{F}| = 5$, $p(A) = 5$.

4. Let $X = \{a, b, c, d, e, f, g\}$,

$$\begin{aligned}\mathcal{F} = \{&\{a, b, c, d, e, f, g\}, && 1\\ &\{a, \quad c, d, e, f, g\}, && 2\\ &\{\quad b, c, d, e, f, g\}, && 3\\ &\{a, b, c, d, e \qquad\}, && 4\\ &\{\qquad c, d, e, f, g\}, && 5\\ &\{a, \quad c, d, e \qquad\}, && 6\\ &\{\quad b, c, d, e \qquad\}, && 7\\ &\{\qquad c, \quad e, f, g\}, && 8\\ &\{a, \quad c, d \qquad\}, && 9\\ &\{\qquad c, d, e \qquad\}, && 10\\ &\{\qquad c, d \qquad\}, && 11\\ &\{\qquad c, \quad e \qquad\}, && 12\\ &\{\qquad c \qquad\}\}, && 13\end{aligned}$$

and $A = \{a, f\}$. Then $|X| = 7$, $|\mathcal{F}| = 13$, and $g(A) = 9$.

Translator's notes. See the appendix for an example satisfying $|X| = 9$, $|\mathcal{F}| = 24$, and $m(A) = 2n(A) = 36$. The following section on the convex hull operator was translated without verifying its mathematical correctness. Since many results were stated without proof, it may contain mathematical errors.

Convex hull. An important example of a closure operator is the convex hull operator $h = conv$ in n-dimensional Euclidean space E^n. We consider the semigroups (1)−(9) for this special case.

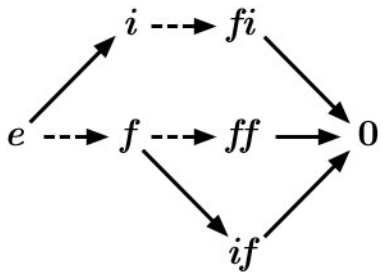
Because of the known relation $hchch = chch$ **[5]**, diagrams (1), (2), and (3) become:

$$chchc \longleftarrow hchc \longleftarrow chc \longleftarrow hc \longleftarrow c \rightleftarrows e \longrightarrow h \longrightarrow ch \longrightarrow hch \longrightarrow chch; \tag{1'}$$

$$cicic \longleftarrow icic \longleftarrow cic \longleftarrow ic \longleftarrow c \rightleftarrows e \longrightarrow i \longrightarrow ci \longrightarrow ici \longrightarrow cici; \tag{2'}$$

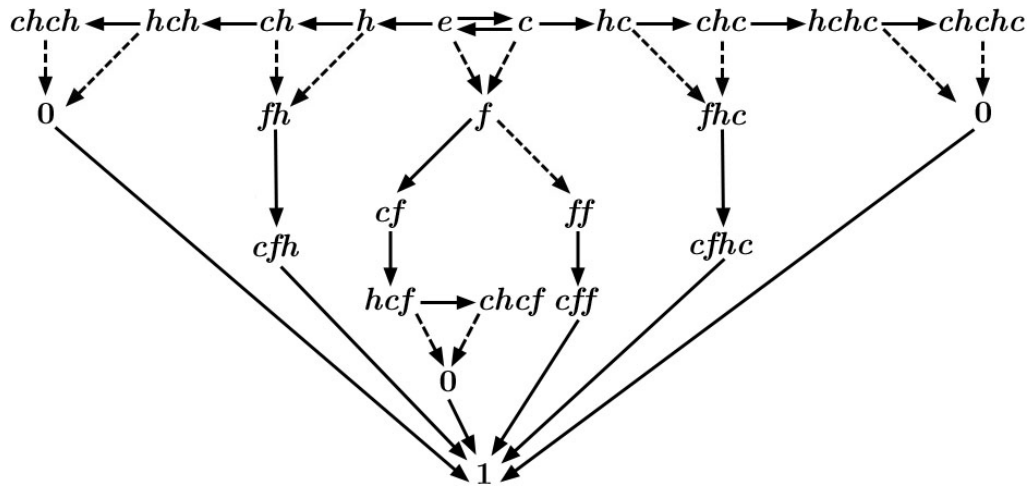
$$hi \longleftarrow i \longleftarrow e \longrightarrow h \longrightarrow ih. \tag{3'}$$

The semigroups in diagrams (4) and (5) remain the same. Going through all the operators in (6), we observe that $fifA = \varnothing$. Consequently, diagram (6) takes the following form:



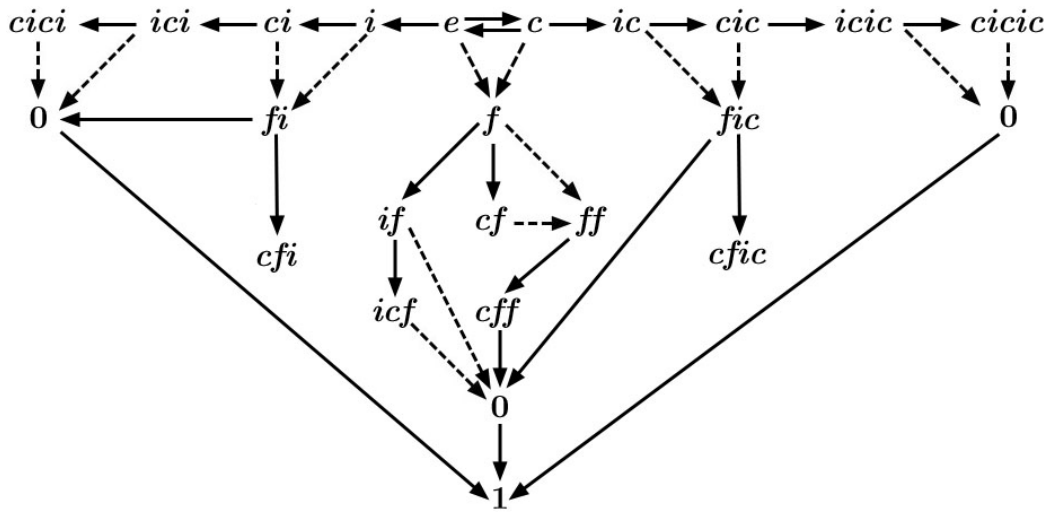
(6')

Similarly, it is easy to verify that diagram (7) takes the following form:

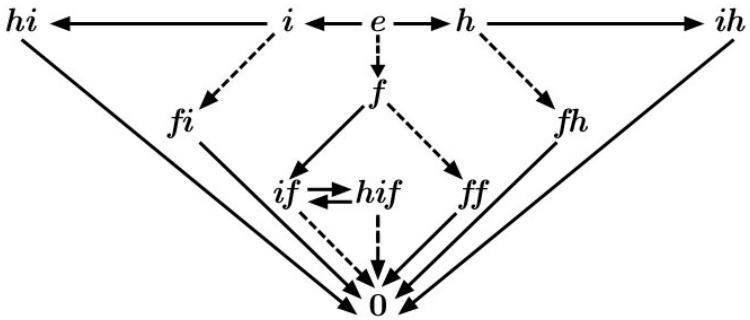


(7′)

Hence, we also obtain new diagrams for (8) and (9):



(8′)



(9′)

The next table shows all possible combinations of values of $k(A)$, $l(A)$, $z(A)$, $p(A)$, $q(A)$, $m(A)$, $n(A)$ for the operator $h = conv$ in the space E^n when $n \geq 2$.

$k(A)$	2	2	2	4	4	4	4	4	6	6	6	6	6	8	8	8	10
$l(A)$	1	1	1	2	2	2	2	2	3	3	3	3	3	4	4	4	5
$z(A)$	2	2	4	2	2	4	4	4	4	4	4	6	6	4	4	6	4
$p(A)$	1	2	2	1	4	2	3	4	2	4	4	3	4	4	4	3	4
$q(A)$	1	2	2	2	3	4	3	3	3	4	4	3	4	4	5	5	5
$m(A)$	2	2	4	4	4	8	4	8	6	6	10	6	6	14	14	10	18
$n(A)$	1	2	2	2	4	4	3	4	3	5	5	3	5	5	7	5	9

We note that the following relationships hold for $h = conv$:

$$k(A) = 2l(A), \quad n(A) \leq m(A) \leq 2n(A), \quad n(A) < 2q(A).$$

Translator's note. The inequality $m(A) \leq 2n(A)$ is violated in the fourth column from the right. The value of 14 in that column may be incorrect.

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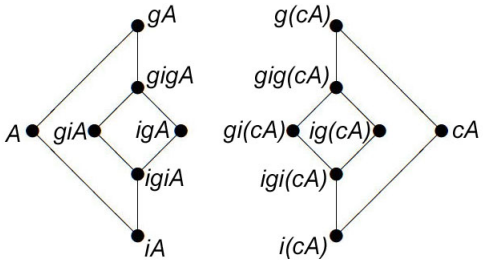
Appendix

The following proofs did not appear in the original paper.

Theorem 2 part 1. If the closure space X contains a set A such that $k(A) = 14$, then $|X| \geq 6$.

PROOF. Suppose $A \subset X$ satisfies $k(A) = 14$. Then the 14 sets below, ordered by inclusion, are distinct.



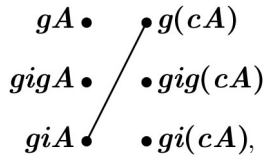
Since the inclusions are strict, we have $|gA| \geq |iA| + 4$ and $|g(cA)| \geq |i(cA)| + 4$. Hence $|X| \geq 4$. But $|X| = 4 \Rightarrow iA = i(cA) = \varnothing$, contradicting $k(A) = 14$. Hence $|X| \geq 5$. Suppose $|X| = 5$. If iA and $i(cA)$ are both nonempty, we get $|gA| \geq |X|$ and $|g(cA)| \geq |X|$, contradicting $gA \neq g(cA)$. We can therefore assume without loss of generality that: $X = \{a, b, c, d, e\}$, $iA = \varnothing$, $igiA = \{a\}$, $giA = \{a, b\}$, $igA = \{a, c\}$, $gigA = \{a, b, c\}$, and $gA = \{a, b, c, d\}$. Note that $giA \cap c(igA) = \{b\}$. Since $giA \cap c(igA)$ is closed, this implies $\{a, b\} = giA = g\varnothing \subset \{b\}$, which is impossible. Conclude $|X| \geq 6$. □

Theorem 3 part 2. When $l(A) = 7$, all sets in (12) are distinct.

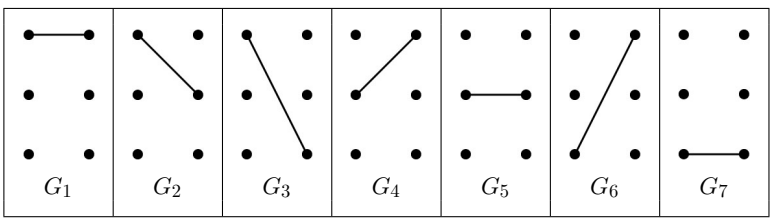
PROOF. The sets in (12) will be labeled:

$$G_1 = gA \cap g(cA), \quad G_2 = gA \cap gig(cA), \quad G_3 = gA \cap gi(cA),$$
$$G_4 = gigA \cap g(cA), \quad G_5 = gigA \cap gig(cA), \quad G_6 = giA \cap g(cA), \quad G_7 = giA \cap gi(cA).$$

It will be convenient to use a graphical notation as well. For example, the set G_6 will be represented as

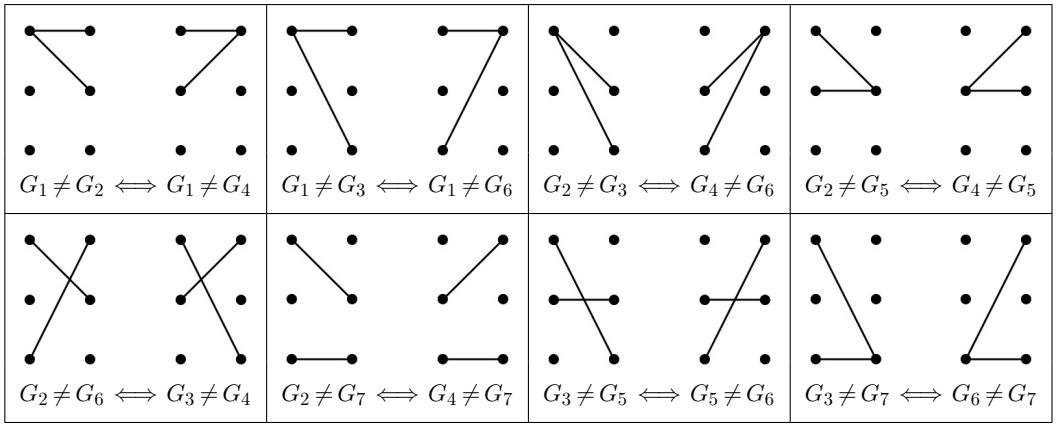


except without the labels. Thus all sets in (12) will be represented as:



(13)

Two simple methods of elimination reduce the number of inequations to prove from $\binom{7}{2} = 21$ to just 5. The first and easiest exploits equivalences based on duality:



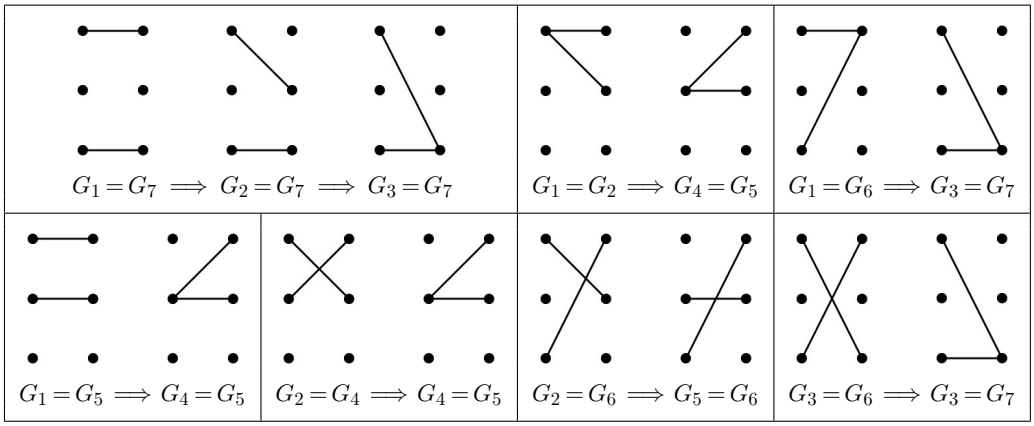
(14)

These equivalences are meant to be interpreted as follows: an inequation holds for all A such that $l(A) = 7$ if and only if its dual inequation (obtained by substituting cA for A) also does. Each equivalence follows from the equivalence $l(A) = 7 \iff l(cA) = 7$.

Assuming in each case that the dual inequation holds for all A such that $l(A) = 7$ (this remains to be proved), table (14) allows us to eliminate the following inequations:

$$G_1 \neq G_3, \quad G_1 \neq G_4, \quad G_2 \neq G_5, \quad G_3 \neq G_4, \quad G_3 \neq G_5, \quad G_4 \neq G_6, \quad G_4 \neq G_7, \quad G_6 \neq G_7.$$

The second elimination method exploits implications based on set inclusion:



(15)

Unlike the situation in table (14) where each equivalence depended on an inequation holding *for all A such that $l(A) = 7$*, each implication in (15) is immediately true for any set A: the consequent equation is obtained by intersecting both sides of the antecedent equation with some set that is represented by a single vertex in the graph. With this in mind, the implications are easy to verify.

Assuming that no consequent equation in (15) holds for any A such that $l(A) = 7$ (this remains to be proved), we can eliminate the following inequations:

$$G_1 \neq G_2, \quad G_1 \neq G_5, \quad G_1 \neq G_6, \quad G_1 \neq G_7, \quad G_2 \neq G_4, \quad G_2 \neq G_6, \quad G_2 \neq G_7, \quad G_3 \neq G_6.$$

It remains to prove the following five inequations for all A such that $l(A) = 7$:

$$G_2 \neq G_3, \quad G_3 \neq G_7, \quad G_4 \neq G_5, \quad G_5 \neq G_6, \quad G_5 \neq G_7.$$

Since we have both $l(A) = 7$ and $l(cA) = 7$, there exist points

$$\begin{aligned}
(i) \quad & x \in gA \setminus gigA = gA \cap igi(cA) \subset gi(cA), \\
(ii) \quad & y \in g(cA) \setminus gig(cA) = g(cA) \cap igiA \subset giA \subset gigA, \text{ and} \\
(iii) \quad & z \in gig(cA) \setminus gi(cA) = gig(cA) \cap igA \subset gigA \subset gA.
\end{aligned}$$

Hence,

$$\begin{aligned}
gA \cap gig(cA) &\neq gA \cap gi(cA) && (G_2 \neq G_3) \text{ by } (iii), \\
gA \cap gi(cA) &\neq giA \cap gi(cA) && (G_3 \neq G_7) \text{ by } (i) \text{ since } x \notin gigA \Longrightarrow x \notin giA, \\
gigA \cap g(cA) &\neq gigA \cap gig(cA) && (G_4 \neq G_5) \text{ by } (ii), \\
gigA \cap gig(cA) &\neq giA \cap g(cA) && (G_5 \neq G_6) \text{ by } (ii), \text{ and} \\
gigA \cap gig(cA) &\neq giA \cap gi(cA) && (G_5 \neq G_7) \text{ by } (iii).
\end{aligned}$$

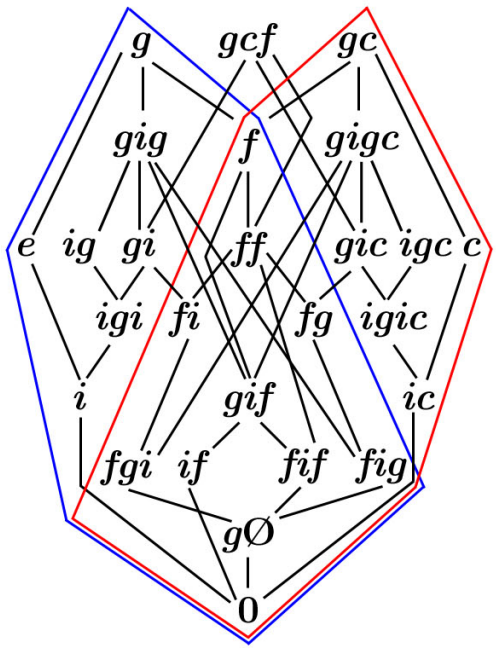
This completes the proof of Theorem 3 part 2.

Theorem 3 part 5. Assuming $h(A) = 9$, the following 13 sets are distinct:

$$X,\ gA,\ gcA,\ fA,\ giA,\ fiA,\ g\varnothing,\ ffA,\ fifA,\ gcfA,\ gifA,\ gcfA \cap gA,\ gcfA \cap gcA.$$

Proof. The table and Hasse diagram below serve as handy references. Each identity represents a given closed set as the union of two disjoint sets: one is open and the other is closed and frontier-like, if not an actual frontier. The semigroup generated by $\{g, i, f\}$ is outlined in blue in the Hasse diagram. Its dual semigroup is outlined in red. Since g and i are dual operators and f is self-dual, we have $n(A) = 18 \Longleftrightarrow n(cA) = 18$.

set	decomposition
gA	$iA \cup fA$ $igiA \cup G_2$ $igA \cup fgA$
$gigA$	$iA \cup G_4$ $igiA \cup G_5$ $igA \cup figA$
giA	$iA \cup fiA$ $igiA \cup fgiA$
$gcfA$	$[iA \cup i(cA)] \cup ffA$ $iA \cup (gcfA \cap gcA)$ $igiA \cup [gcfA \cap gig(cA)]$
$gcfA \cap gA$	$iA \cup ffA$
$gcfA \cap gigA$	$iA \cup (ffA \cap gigA)$
fA	$ifA \cup ffA$
$gifA$	$ifA \cup fifA$



(16)

Let us first dispense with X. Note that every set in the list except X and $gcfA$ is a subset of either gA or gcA. Since $gA \neq X$ and $gcA \neq X$, none of these sets equals X. By assumption $ifA \neq \varnothing$, hence $gcfA = c(ifA) \neq X$. Therefore X is distinct from the other 12 sets in the list.

DEFINITION. Sets A and B satisfying $i(A \setminus B) \neq \varnothing$ will be called *interior-distinct*. We also use this term to describe families $\mathcal{S}$ and $\mathcal{T}$ such that S and T are interior-distinct for all $(S,T) \in \mathcal{S} \times \mathcal{T}$.

CLAIM 1. The following families are pairwise interior-distinct:

$$\begin{aligned}\mathcal{S} &= \{gA, giA, gcfA \cap gA\},\\ \mathcal{T} &= \{gcA, gcfA \cap gcA\},\\ \mathcal{U} &= \{gcfA\},\\ \mathcal{V} &= \{fA, fiA, g\varnothing, ffA, fifA, gifA\}.\end{aligned}$$

PROOF. Note that every set in $\mathcal{V}$ is contained in fA and we have $fA \cap [iA \cup i(cA)] = \varnothing$. We also have $iA \cup i(cA) \subset gcfA$. Hence, for any $S \in \mathcal{S}$, $T \in \mathcal{T}$, $U \in \mathcal{U}$, and $V \in \mathcal{V}$, we have $iA \subset i(S \setminus T)$, $i(cA) \subset i(U \setminus S)$, $iA \subset i(S \setminus V)$, $iA \subset i(U \setminus T)$, $i(cA) \subset i(T \setminus V)$, and $iA \cup i(cA) \subset i(U \setminus V)$. This proves Claim 1.

Since all 12 remaining sets appear in the four families above, it remains to show that no sets within a given family can equal each other.

CLAIM 2. Any two sets within $\mathcal{S}$, or within $\mathcal{T}$, are distinct.

PROOF. From (16) we see that if $gA = giA$, then $fA = fiA$, contradicting our assumption. The other three inequations are proved similarly.

CLAIM 3. All sets in $\mathcal{V}$ are distinct.

PROOF. Since $h(A) = 9$, we need only show that $gifA$ is distinct from the other five sets. Since $ifA \neq \varnothing$ by assumption, we have $g\varnothing \subsetneqq fifA \subsetneqq gifA$. Since we also have $fifA \subsetneqq ffA \subsetneqq fA$ and $ifA \cap ffA = \varnothing$, there exists $x \in ffA \setminus fifA = ffA \setminus gifA \subset fA \setminus gifA$. Thus $gifA \neq ffA$ and $gifA \neq fA$. Since $fiA \subset ffA$, we have $fiA \cap ifA = \varnothing$. Hence, there exists a point $x \in gifA \setminus fiA$. Thus $gifA \neq fiA$. This proves Claim 3.

This completes the proof of Theorem 3 part 5.

Theorem 3 part 7. If $n(A) = 18$, then $|\mathcal{F}| \geq 24$.

PROOF. Define

$$\begin{aligned}G_8 &= gigA \cap gi(cA) = figA,\\ G_9 &= giA \cap gig(cA) = fgiA.\end{aligned}$$

CLAIM 4. The following families are pairwise interior-distinct:

$$\begin{aligned}\mathcal{W} &= \{gA, gigA, giA, gcfA \cap gA, gcfA \cap gigA\},\\ \mathcal{X} &= \{g(cA), gig(cA), gi(cA), gcfA \cap gcA, gcfA \cap gig(cA)\},\\ \mathcal{Y} &= \{gcfA\},\\ \mathcal{Z} &= \{G_1, G_2, G_3, G_4, G_5, G_6, G_8, G_9, g\varnothing, ffA, fifA, gifA, ffA \cap gigA, ffA \cap gig(cA)\}.\end{aligned}$$

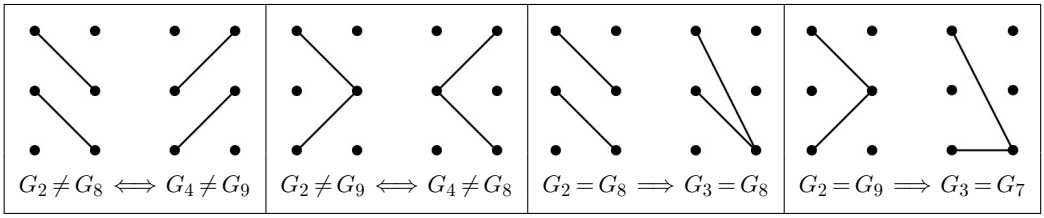
PROOF. Claim 4 holds by the same argument used to prove Claim 1.

Since $n(A) = 18 \Longrightarrow h(A) = 9$ we dispense with the set X just as we did in the proof of Theorem 3 part 5 and conclude that X does not equal any of the sets listed in Claim 4.

CLAIM 5. The first 11 sets in $\mathcal{Z}$ are distinct.

PROOF. The first six sets are distinct since $n(A) = 18 \Longrightarrow l(A) = 7$.

Note, interpreting the diagrams below as in the proof of Theorem 3 part 2, we have the following equivalences and implications:



(17)

We have $G_3 \neq G_7$ (since $l(A) = 7$) and $G_3 \neq G_8$ (since $n(A) = 18$). Thus $G_2 \neq G_8$ and $G_2 \neq G_9$. Hence $G_4 \neq G_8$ and $G_4 \neq G_9$. The point $z \in G_5 \setminus G_7$ used to prove $G_5 \neq G_7$ in part 2 also satisfies $z \in G_5 \setminus G_8$. Hence $G_5 \neq G_8$. We get $G_5 \neq G_9$ by duality.

Recall, $G_5 \subset G_2$ and $G_5 \subset G_4$. Since $g\varnothing \subsetneq fifA \subsetneq gifA \subset G_5$, it follows that neither $g\varnothing$ nor $fifA$ equals G_2, G_4, or G_5.

Note that $G_1 \neq G_2$ implies $ffA \cap igiA = (fA \cap gcfA) \cap igiA = fA \cap (gcfA \cap igiA) = fA \cap giA \neq \varnothing$. Since $G_5 \cap igiA \subset G_2 \cap igiA = \varnothing$, this implies $ffA \neq G_2$ and $ffA \neq G_5$. By a dual argument, $ffA \neq G_4$.

The remaining inequations hold by assumption since $G_1 = fA$, $G_3 = fgA$, $G_6 = fiA$, $G_8 = figA$, and $G_9 = fgiA$. This proves Claim 5.

Claim 6. The first four sets in $\mathcal{W}$ are distinct and the first four sets in $\mathcal{X}$ are distinct.

Proof. This follows from (16), since fA, G_4, fiA, and ffA are distinct by Claim 5.

The proof of Theorem 3 part 7 now breaks into three cases, two of which are dual:

case	condition	new distinct sets
1	$gcfA \cap gig(cA) = gi(cA)$	$gifA$ $gcfA \cap gigA$ $ffA \cap gigA$
2	$gcfA \cap gigA = giA$	$gifA$ $gcfA \cap gig(cA)$ $ffA \cap gig(cA)$
3	$gcfA \cap gig(cA) \neq gi(cA)$ and $gcfA \cap gigA \neq giA$	$gcfA \cap gigA$ $gcfA \cap gig(cA)$ $ffA \cap gigA$ $ffA \cap gig(cA)$

(18)

Case 1. The only inequations that depend on the case assumption are: $gifA \neq G_5$, $gcfA \cap gigA \neq giA$, $ffA \cap gigA \neq G_6$. We prove these first.

Suppose $gifA = G_5$. Then

$$figA = gigA \cap gi(cA) = gigA \cap [gcfA \cap gig(cA)] = gcfA \cap G_5 = gcfA \cap gifA = fifA.$$

Since this contradicts $n(A) = 18$, we get $gifA \neq G_5$.

Suppose $gcfA \cap gigA = giA$. Then

$$G_8 = gigA \cap gi(cA) = gigA \cap [gcfA \cap gig(cA)] = giA \cap gig(cA) = G_9,$$

contradicting Claim 5. Hence $gcfA \cap gigA \neq giA$. Since $G_6 = fiA$, it follows immediately from (16) that $ffA \cap gigA \neq G_6$.

As we proceed to show, the remaining inequations in Case 1 all hold without the case assumption.

Since $G_5 \underset{\neq}{\subset} G_2$, $G_5 \underset{\neq}{\subset} G_4$, and $gifA \subset G_5$, we get $gifA \neq G_2$ and $gifA \neq G_4$. That $gifA$ does not equal any of its other eight predecessors in $\mathcal{Z}$ follows immediately from the condition $n(A) = 18$.

We have $gcfA \cap gigA \neq gA$, since otherwise $gigA \subset gA = gcfA \cap gigA \subset gigA$, contradicting the inequation $gigA \neq gA$. We have $gcfA \cap gigA \neq gigA$, since otherwise

$$gcfA = igi(cA) \cup (gcfA \cap gigA) = igi(cA) \cup gigA = X.$$

It is evident from (16) that the equation $gcfA \cap gigA = gcfA \cap gA$ implies $i(cA) = igi(cA)$, contradicting $gA \neq gigA$. Hence $gcfA \cap gigA \neq gcfA \cap gA$. Thus $gcfA \cap gigA$ equals none of its predecessors in $\mathcal{W}$.

It follows from (16) and the three inequations proved in the previous paragraph that the set $ffA \cap gigA$ is distinct from $fA\,(=G_1)$, G_4, and ffA. We get $ffA \cap gigA \neq G_2$ and $ffA \cap gigA \neq G_5$ by the same argument used to prove $ffA \neq G_2$ and $ffA \neq G_5$ in Claim 5. If $ffA \cap gigA = G_3$, then $gA = igA \cup G_3 = igA \cup (ffA \cap gigA) \subset gigA$, contradicting $gA \neq gigA$. Hence $ffA \cap gigA \neq G_3$.

Suppose $fiA \subset fgA$. Then $G_6 = giA \cap gcA = fiA \subset fgA = gA \cap gi(cA) \subset gi(cA)$. Hence $G_6 = giA \cap G_6 \subset giA \cap gi(cA) = G_7$. Since $G_7 \subset G_6$ and $l(A) = 7$, this contradicts $G_6 \neq G_7$. Hence there exists a point $x \in fiA \setminus fgA$. Since $fiA \subset ffA \cap gigA$, we get $ffA \cap gigA \neq fgA \cap gigA = G_3 \cap gigA = G_8$. We have $G_9 = giA \cap gig(cA) = fgiA \underset{\neq}{\subset} fiA \subset ffA \cap gigA$. Hence $ffA \cap gigA \neq G_9$.

Since $g\varnothing \underset{\neq}{\subset} fifA \subset ffA \cap gigA$, we get $ffA \cap gigA \neq g\varnothing$.

Suppose $ffA \cap gigA = fifA$. This implies:

$$G_4 = fA \cap gigA = (ifA \cup ffA) \cap gigA = (ifA \cap gigA) \cup (ffA \cap gigA) = ifA \cup fifA = gifA \subset G_5.$$

But we have $G_5 \subset G_4$ and $G_4 \neq G_5$. Conclude $ffA \cap gigA \neq fifA$.

Suppose $ffA \cap gigA = ffA$. Then

$$gcfA \cap gigA = iA \cup (ffA \cap gigA) = iA \cup ffA = gcfA \cap gA,$$

contradicting $gcfA \cap gigA \neq gcfA \cap gA$. Hence $ffA \cap gigA \neq ffA$.

Lastly, suppose $ffA \cap gigA = gifA$. Since $ifA \subset gifA \subset gigA$, this implies

$$G_4 = fA \cap gigA = (ifA \cup ffA) \cap gigA = (ifA \cap gigA) \cup (ff \cap gigA) = ifA \cup gifA = gifA,$$

contradicting $gifA \neq G_4$. Hence $ffA \cap gigA \neq gifA$. Conclude $|\mathcal{F}| \geq 24$.

Case 2. This case is obtained by substituting cA for A in Case 1. Since $n(A) = 18 \iff n(cA) = 18$, the proof for Case 1 implies $|\mathcal{F}| \geq 24$ in Case 2.

Case 3. Note, the only inequations within either $\mathcal{W}$ or $\mathcal{X}$ that were not proved independently of the case assumptions in Case 1 and Case 2, are the inequations in the Case 3 assumption. Thus no inequations remain to be proved within either $\mathcal{W}$ or $\mathcal{X}$ in Case 3.

Suppose $ffA \cap gigA = G_6$. This implies that $giA = iA \cup G_6 = iA \cup (ffA \cap gigA) = gcfA \cap gigA$, contradicting the Case 3 assumption. Hence $ffA \cap gigA \neq G_6$. All other inequations involving $ffA \cap gigA$ were proved independently of the case assumption in Case 1. Therefore $|\mathcal{F}| \geq 24$.

This completes the proof of Theorem 3 part 7.

Note. By duality the set $ffA \cap gig(cA)$ is also new above. Hence in Case 3 we have $|\mathcal{F}| \geq 25$.

The next two pages give an example showing that the inequality $|\mathcal{F}| \geq 24$ is sharp.

The following example displays a closure space X and subset A such that $|\mathcal{F}| = 24$ and $n(A) = 18$. Let $X = \{a, b, c, d, e, f, g, h, i\}$. Let $\mathcal{F}$ be generated by the family:

$$\begin{array}{l}
\{\{a,b,c,d,e,f,g,h,i\},\\
\{a,b,c,\ \ e,f,g,h,i\},\\
\{a,b,c,d,e,f,g,h\ \ \},\\
\{a,b,c,\ \ e,\ \ g,h,i\},\\
\{\ \ b,c,d,e,f,g,\ \ i\},\\
\{a,\ \ c,d,e,f,g,h\ \ \},\\
\{a,\ \ c,\ \ \ \ \ g,h\ \ \},\\
\{\ \ \ \ \ d,e,f,g\ \ \ \ \}\}.
\end{array}$$

Thus,

$$\begin{array}{rll}
\mathcal{F} = \{\{a,b,c,d,e,f,g,h,i\}, & & 1\\
\{a,b,c,\ \ e,f,g,h,i\}, & & 2\\
\{a,b,c,d,e,f,g,h\ \ \}, & & 3\\
\{a,b,c,\ \ e,\ \ g,h,i\}, & & 4\\
\{\ \ b,c,d,e,f,g,\ \ i\}, & & 5\\
\{a,\ \ c,d,e,f,g,h\ \ \}, & & 6\\
\{a,b,c,\ \ e,f,g,h\ \ \}, & & 7\\
\{\ \ b,c,\ \ e,f,g,\ \ i\}, & & 8\\
\{a,\ \ c,\ \ e,f,g,h\ \ \}, & & 9\\
\{a,b,c,\ \ e,\ \ g,h\ \ \}, & & 10\\
\{\ \ b,c,d,e,f,g\ \ \ \ \}, & & 11\\
\{\ \ b,c,\ \ e,\ \ g,\ \ i\}, & & 12\\
\{a,\ \ c,\ \ e,\ \ g,h\ \ \}, & & 13\\
\{\ \ \ \ c,d,e,f,g\ \ \ \ \}, & & 14\\
\{\ \ b,c,\ \ e,f,g\ \ \ \ \}, & & 15\\
\{a,\ \ c,\ \ \ \ \ g,h\ \ \}, & & 16\\
\{\ \ \ \ \ d,e,f,g\ \ \ \ \}, & & 17\\
\{\ \ \ \ c,\ \ e,f,g\ \ \ \ \}, & & 18\\
\{\ \ b,c,\ \ e,\ \ g\ \ \ \ \}, & & 19\\
\{\ \ \ \ \ \ \ e,f,g\ \ \ \ \}, & & 20\\
\{\ \ \ \ c,\ \ e,\ \ g\ \ \ \ \}, & & 21\\
\{\ \ \ \ \ \ \ e,\ \ g\ \ \ \ \}, & & 22\\
\{\ \ \ \ c,\ \ \ \ \ g\ \ \ \ \}, & & 23\\
\{\ \ \ \ \ \ \ \ \ g\ \ \ \ \}\}. & & 24
\end{array}$$

Let $A = \{a, b, c, d\}$. The family generated by A under the operations $\{g, i, f\}$ contains 18 distinct sets:

$$
\begin{aligned}
gA &= \{a, b, c, d, e, f, g, h\ \}, && 1\\
gigA &= \{a,\ \ c, d, e, f, g, h\ \}, && 2\\
fA &= \{a, b, c,\ \ e, f, g, h\ \}, && 3\\
ffA &= \{\ \ b, c,\ \ e, f, g\ \ \}, && 4\\
gifA &= \{a,\ \ c,\ \ \ \ g, h\ \}, && 5\\
igA &= \{a,\ \ \ d,\ \ f,\ \ h\ \}, && 6\\
giA &= \{\ \ \ \ d, e, f, g\ \ \}, && 7\\
fgA &= \{\ \ b, c,\ \ e,\ \ g\ \ \}, && 8\\
A &= \{a, b, c, d\ \ \ \ \ \}, && 9\\
fiA &= \{\ \ \ \ \ e, f, g\ \ \}, && 10\\
figA &= \{\ \ \ \ c,\ \ e,\ \ g\ \ \}, && 11\\
if &= \{a,\ \ \ \ \ \ \ \ h\ \}, && 12\\
fgiA &= \{\ \ \ \ \ e,\ \ g\ \ \}, && 13\\
fifA &= \{\ \ \ \ c,\ \ \ \ g\ \ \}, && 14\\
igiA &= \{\ \ \ \ d,\ \ f\ \ \ \ \}, && 15\\
iA &= \{\ \ \ \ d\ \ \ \ \ \}, && 16\\
g\varnothing &= \{\ \ \ \ \ \ \ g\ \ \}, && 17\\
\varnothing &= \{\ \ \ \ \ \ \ \ \ \}. && 18
\end{aligned}
$$

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