# The Kuratowski Closure-Complement Theorem

By Greg Strabel

The Kuratowski Closure-Complement Theorem, a result of basic point-set topology, was first posed and proven by the Polish mathematician Kazimierz Kuratowski in 1922. Since then, Kuratowski's Theorem and its related results, in particular, the structure of the Kuratowski monoid of a topological space, have been the subject of a plethora of papers. The formal statement of the theorem is as follows:

**Theorem 1: The Kuratowski Closure-Complement Theorem:** *Let $(X, T)$ be a topological space and suppose $A \subseteq X$. Then there are at most 14 distinct sets that can be formed by taking complements and closures of A. Moreover, this bound is attained for a subset of $\Re$ with the standard topology.*

As in the proof of Gardner and Jackson, we make use of operator notation. Given a topological space $(X, T)$ define the complement operator $a$ and the closure operator $b$ on subsets $A \subseteq X$ by $a(A) = X \setminus A$ and $b(A) = closure(A)$. Notice that for any topological space $(X, T)$ and subset $A \subseteq X$, $aa(A) = A$. Then given any topological space $(X, T)$, the set of all distinct operators on $(X, T)$ produced by compositions of elements of the set $\{a, b\}$ forms a monoid with identity element $aa$. This monoid is referred to as the Kuratowski monoid on $(X, T)$.

For any topological space $(X, T)$, there is a natural partial order on the Kuratowski monoid on $(X, T)$. If $o_1$ and $o_2$ are elements of the Kuratowski monoid on $(X, T)$, we define the partial order $\leq$ as $o_1 \leq o_2$ if for every $A \subseteq X$, $o_1(A) \subseteq o_2(A)$. We are now ready to prove Theorem 1.

**Proof:** Let $(X, T)$ be a topological space. We have already seen that $aa = id$. Note also that $bb = b$. This immediately implies that any operator of the Kuratowski monoid on $(X,$

*T)* is equivalent to one of the following: *id*, *a*, *b*, *ab*, *ba*, *aba*, *bab*, *abab*, *baba*, *ababa*, *babab*, *abab…ab*, *baba…ba*, *abab…aba* or *baba…bab*.

We now show that *bab* = *bababab*. First note that $aba(A) = ab(X \setminus A) = a(interior(X \setminus A) \cup \partial A) = interior(A)$. Then *ababab* ≤ *bab* since *ababab(A)* is the interior of *bab(A)*. Since *bb* = *b*, it follows that *bababab* ≤ *bbab* = *bab*. Also, *abab* ≤ *b* since *abab (A)* is the interior of *b(A)*. Hence, *babab* ≤ *bb* = *b*. Then *ababab* ≥ *ab*, and therefore *bababab* ≥ *bab*. We conclude that *bab* = *bababab*. From this it is clear that any word on $\{a, b\}$ that has length greater than 7 must be equivalent to a word of length at most 7.

Thus, for any topological space *(X, T)*, each operator in the Kuratowski monoid on *(X, T)* is equivalent to at least one of the following:

*id*, *a*, *b*, *ab*, *ba*, *aba*, *bab*, *abab*, *baba*, *ababa*, *babab*, *ababab*, *bababa*, *abababa*

Therefore, given any topological space *(X, T)*, the Kuratowski monoid on *(X, T)* has order at most 14 and hence for any $A \subseteq X$, there are at most 14 distinct sets that can by produced by taking closures and complements of *A*.

To complete the proof of the theorem we show that the set $A \subseteq \mathfrak{R}$ given by

$$A = (0, 1) \cup (1, 2) \cup \{3\} \cup ([4, 5] \cap \mathbf{Q})$$

attains the bound of 14; that is, we can produce 14 distinct sets from *A* by taking complements and closures. These sets are:

$$id(A) = A = (0, 1) \cup (1, 2) \cup \{3\} \cup ([4, 5] \cap \mathbf{Q})$$

$$a(A) = (-\infty, 0] \cup \{1\} \cup [2, 3) \cup (3, 4) \cup ([4, 5] \cap \mathbf{I}) \cup (5, -\infty)$$

$$b(A) = [0, 2] \cup \{3\} \cup [4, 5]$$

$$ab(A) = (-\infty, 0) \cup (2, 3) \cup (3, 4) \cup (5, -\infty)$$

$$ba(A) = (-\infty, 0] \cup \{1\} \cup [2, -\infty)$$

$$aba(A) = (0, 1) \cup (1, 2)$$

$$bab(A) = (-\infty, 0] \cup [2, 4] \cup [5, -\infty)$$

$$abab(A) = (0, 2) \cup (4, 5)$$

$$baba(A) = [0, 2]$$

$$ababa(A) = (-\infty, 0) \cup (2, -\infty)$$

$$babab(A) = [0, 2] \cup [4, 5]$$

$$ababab(A) = (-\infty, 0) \cup (2, 4) \cup (5, -\infty)$$

$$bababa(A) = (-\infty, 0] \cup [2, -\infty)$$

$$abababa(A) = (0, 2)$$

□

An equivalent statement of the Kuratowski Closure-Complement Theorem is the following:

**Theorem 2:** *Let (X, T) be a topological space and suppose $A \subseteq X$. Then there are at most 7 distinct sets that can be formed by taking closures and interiors of A. Moreover, this bound is attained for a subset of $\Re$ with the standard topology.*

The proof of this theorem follows immediately from the proof of Theorem 1. First, if *(X, T)* is the empty space then there is obviously only one subset of *X*, the empty set, and we can only form one set by taking closures and interiors, the empty set. Suppose *(X, T)* is not the empty space. Notice that because for any non-empty topological space *(X, T)* and any subset $A \subseteq X$, $X$ can be written as the disjoint union $X = \operatorname{int}(X \setminus A) \cup \partial A \cup \operatorname{int}(A)$, two words on $\{a, b\}$ cannot be equivalent if one contains an odd number of complements while the other contains an even number of complements. Thus, we can partition the Kuratowski monoid on *(X, T)* into operators with an even number of complements and operators with an odd number of complements; we call these sets of operators the *even Kuratowski operators on (X, T)* and the *odd Kuratowski operators on (X, T)*, respectively. Since *aba* is the interior operator, we see that the set of all possibly distinct even Kuratowski operators, *id*, *b*, *aba*, *abab*, *baba*, *babab*, and *abababa*, corresponds exactly to the set of all possibly distinct operators formed by composition of closures and interiors. If we let *i* denote the interior operator, the possibly distinct even Kuratowski operators become *id*, *b*, *i*, *ib*, *bi*, *bib*, and *ibi*, establishing the fact that there are at most seven distinct operators formed by compositions of closures and interiors. Of course, the same subset $A \subseteq \Re$ from the proof of Theorem 1 can be used again here to show that the bound is attained. Theorem 1 then follows from Theorem 2 by the fact that distinct odd Kuratowski operators arise as the composition of distinct even Kuratowski operators with $a$. To see this, suppose we have $n \leq 7$ distinct even Kuratowski operators on $(X, T)$ and suppose that $o_1$ and $o_2$ are two distinct even operators. Then $o_1a \neq o_2a$ , else $o_1 = o_1aa = o_2aa = o_2$. Hence there are at least $2n$ Kuratowski operators on $(X, T)$. That there are not more follows from the fact that if $o$ is an odd operator then $oa$ is an even operator and $o = (oa)a$.

Having proven Kuratowski's Closure-Complement Theorem, we naturally wonder if we can actually put the result to work in discovering any other interesting properties of a topological space. The answer to this question happens to be yes. The following definition will aid in our discussion.

**Definition:** *Let $(X, T)$ be a topological space and $A \subseteq X$.*

*(i)* *$k(A)$ (the k-number of A) denotes the number of distinct sets that can be obtained from A by taking closures and complements. We call a set with k-number n an n-set.*

*(ii)* *$k((X, T))$ (the k-number of $(X, T)$) denotes $\max\{k(A): A \subseteq X\}$*

*(iii)* *$K((X, T))$ (the K-number of $(X, T)$) denotes the number of distinct Kuratowski operators on $(X, T)$; that is, the order of the Kuratowski monoid on $(X, T)$.*

It is clear from this definition that for any topological space $(X, T)$ and subset $A \subseteq X$ ,

$$1 \leq k(A) \leq k((X, T)) \leq K((X, T)) \leq 14.$$

The next result is due to Chagrov.

**Theorem 3:** *Given a non-empty topological space (X, T), K((X, T))∈{2, 6, 8, 10, 14}. Moreover, for each n∈{2, 6, 8, 10, 14}, there exists a topological space (X, T) such that K((X, T)) = n. There are a total of six distinct Kuratowski monoids, one each of orders 2, 6, 8 and 14 and two of order 10.*

**Outline of Proof:** The fact that Kuratowski monoids of non-empty topological spaces always have even order was proven in our discussion of Theorem 2, so this part of the result should come as no surprise. To understand the rest of Theorem 3, we start by considering $\Re$ with the standard topology. Our proof of Theorem 1 showed that the K-number of $\Re$ is 14. If we partition the Kuratowski monoid on $\Re$ into even and odd operators we see, by Theorem 2, that the set of distinct even operators is exactly the set *id*, *b*, *i*, *ib*, *bi*, *bib*, and *ibi*. Figure 1.1 of Gardner and Jackson depicts the partial order on the even Kuratowski operators on $\Re$.

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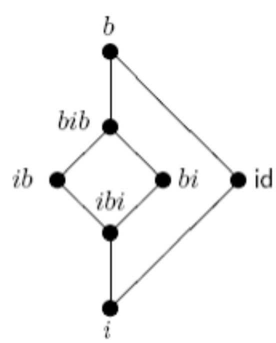


FIGURE 1.1. The seven different operators using $b$ and $i$.

The partial order on the odd Kuratowski operators on $\Re$ can be obtained from the figure by applying the complement operator to the right side of each of the operators in the diagram. A space with K-number 14, such as $\Re$ with the standard topology, is called a *Kuratowski space*. The previous discussion of the Kuratowksi monoid on $\Re$ clearly holds for the Kuratowski monoid of any Kuratowski space.

Chagrov's proof of Theorem 3 shows that there are six distinct Kuratowski monoids, one each of orders 2, 6, 8 and 14 and two of order 10. It should be clear from our discussion of Theorem 2 that all possibly distinct Kuratowski monoids can be derived as collapses of Figure 1.1 and the corresponding collapses of the partial order on the odd Kuratowski operators. First, notice that if $o_1, o_2$ and $o_3$ are three Kuratowski operators such that $o_1 \le o_2 \le o_3$ and $o_1 = o_3$, then $o_1 = o_2 = o_3$. Also, notice that for any operator $o \in \{b, bib, bi, ib, ibi, i\}$, if $o = id$ then for each $o' \in \{b, bib, bi, ib, ibi, i\}$, $o' = id$, that is, Figure 1.1 collapses to a single point, *id*. To see this, observe that if there exists $o \in \{b, bib, bi, ib, ibi, i\}$ such that $o = id$ then either every subset of $X$ is open or every subset of $X$ is closed. In either case, this implies that $T$ is the discrete topology so that there is only one even operator, *id*, and one odd operator, *a*, in the Kuratowski monoid on $(X, T)$. Finally, notice that for each $o \in \{b, bib, bi, ib, ibi, i\}$, $o$ a $aoa$ is an automorphism that replaces $i$ by $b$ and replaces $b$ by $i$. From this, we see that any collapse of Figure 1.1 must preserve the symmetry of the diagram. These observations greatly reduce the number of possible collapses of Figure 1.1 that we must consider. We now go case by case considering collapses of Figure 1.1.

**Case 1:** $ibi = bi$. That is, $abababa = baba$. Hence $aibia = abia$, which gives $babab = aababab aa = ababaa = abab$. Thus $bib = ib$. Since $bib \ge ibi$, this gives the second diagram in Figure 2.1 of Gardner and Jackson.

Example: The cofinite topology on $\Re$.

**Case 2:** $ibi = ib$. That is, $abababa = abab$. Hence $aibia = aiba$, which gives $babab = aabababaa = aababa = baba$. Thus $bib = bi$. Since $bib \ge ibi$, this gives the first diagram in Figure 2.1 of Gardner and Jackson.

Example: Let $X = \Re$ and let $T = \{O \subset \Re : 0 \notin O\} \cup \{\Re\}$.

**Case 3:** $ib = bi$. This implies $bib = bbi = bi = bii = ibi = iib = ib$, giving the second diagram in Figure 2.1 of Gardner and Jackson.

Example: Let $X = \{a, b, c, d, e\}$, let $S = \{\{a\}, \{b\}, \{a, c\}, \{b, d\}, \{a, c, e\}, \{b, d, e\}\}$ and let $T$ be the topology generated by the subbasis $S$.

**Case 4:** $ibi = i$. That this implies $bib = b = ib$ and $bi = i$ and therefore corresponds to the fourth diagram in Figure 2.1 of Gardner and Jackson requires a more in depth argument than the previous cases. Actually, $bib = b$ is trivial; it follows immediately from the fact that $ibi = i$, which implies that $aibia = aia$ and therefore $bib = b$ from our discussion of the automorphism $o \mapsto aoa$. To prove that $ib = b$ and $bi = i$, all we need is to prove that $ib = b$ from which it follows again from the automorphism argument that $bi = i$. To prove that $ib = b,$ it suffices to show that every closed set is open.

For each $x \in X$, let $S_x = i(X \setminus \{x\})$. Then because $ibi = i$, we have $S_x = i(S_x) = ibi(S_x)$.

Since $bib = b$, $b(S_x) \neq X$, else $bi(S_x) = b(S_x) = X$, which implies that $S_x = ibi(S_x) = i(X) = X$, a contradiction. Hence $x \notin b(S_x)$.

Now for each $x \in X$, let $G_x = X \setminus b(S_x)$ so that $x \in G_x$ and $G_x$ is open in $X$. Suppose $y \in G_x$ and let $H$ be an open set containing $y$. Then $H \not\subset X \setminus G_x = b(S_x)$, which implies that $H \not\subset ib(S_x) = ibi(S_x) = i(S_x) = S_x$. Because $S_x$ contains all open sets that do not contain $x$, we conclude that $H$ contains $x$. Hence, every open set containing $y$ also contains $x$ so that $y \in b(\{x\})$. Since $y$ is an arbitrary element of $G_x$, we conclude that $b(G_x) \subseteq b(\{x\}) \subseteq b(G_x)$ and therefore, $b(\{x\}) = b(G_x)$.

Now let $K$ be a closed set in $X$ containing an element $x$. Then $b(\{x\}) \subseteq K$ so that $x \in G_x \subseteq b(G_x) = b(\{x\}) \subseteq K$. Thus, $K$ can be written as a union of open sets $G_x$. We have proven that every closed set in $X$ is open. This completes the proof of case 4.

Example: Let $X = \boldsymbol{Z}$, let $B = \{\{evens\}, \{odds\}\}$ and let $T$ be the topology generated by the basis $B$.

From our automorphism argument, it is clear that between these four cases and the cases of the discrete space and the Kuratowski space, we have exhausted all possible collapses of Figure 1.1 representing Kuratowski monoids of a topological space. (There are in fact two other possible collapses of Figure 1.1, but the rigorous proof of Case 4 actually disqualifies these from being Kuratowski monoids of a topological space, since in the case that *ibi* = *i*, we must also have *ib* = *b* and *bi* = *i*.)

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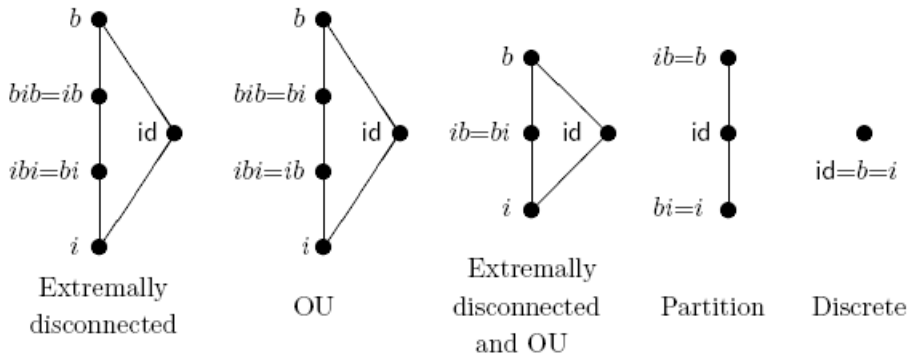


FIGURE 2.1. Orderings on 5 possible Kuratowski monoids.

We now introduce additional definitions from Gardner and Jackson which will prove useful.

**Definition:**

*(i)* *A space is* extremally disconnected *if the closure of any open set is open.*

(ii) *A space is* resolvable *if it contains a dense set with empty interior.*

(iii) *A space is* open unresolvable *if no open subspace is resolvable.*

(iv) *A space is a* partition space *if its open sets form a Boolean algebra.*

From this definition and the discussion of the proof of Theorem 3 we arrive at the following theorem.

**Theorem 4:**

*(i)* *A space satisfies ibi = bi if and only if it is extremally disconnected.*

*(ii)* *A space satisfies ibi = ib if and only if it is open unresolvable if and only if each of its dense sets has a dense interior.*

*(iii)* *A space satisfies ib = bi if and only if it is extremally disconnected and open unresolvable.*

*(iv)* *A space satisfies ib = b if and only if it is a partition space if and only if it's open sets are clopen.*

*(v)* *A space satisfies i = b if and only if it is discrete.*

**Proof:**

Part (i) is trivially true. If *ibi = bi*, then for each open set *O*, *ib(O) = ibi(O) = bi(O) = b (O)*, so that the closure of *O* is open and *(X, T)* is extremally disconnected. If *(X, T)* is extremally disconnected, then for each $A \subseteq X$, *i(A)* is open so that *bi(A)* is open and therefore *ibi(A) = bi(A)*, from which we conclude *ibi = bi*.

Part (ii) requires some work. Aull proves that open unresolvable spaces are exactly those in which every dense set has dense interior and that such spaces satisfy *ibi = ib*. To show the only if direction suppose *(X, T)* satisifes *ibi = ib* and let *A* be dense in *X*. Then *ib(A) = i(X) = X*. Because *ibi(A) = ib(A) = X*, *bi(A) = X* so that *i(A)* is dense in *X* as well.

Part (iii) follows immediately from parts (i) and (ii). If *ib = bi* then we also have *ibi = bii = bi* and *ibi = iib = ib*. If *ibi = ib* and *ibi = bi* then clearly *ib = ibi = bi*.

Part (iv) follows from basic facts about Boolean algebras. Clearly the open sets of a space *(X, T)* form a Boolean algebra if and only if its open sets are clopen. Suppose the open sets of the space *(X, T)* are clopen, then for any $A \subseteq X$, *b(A)* is clopen so that *ib(A) = b (A)* and therefore *ib = b*. Suppose *ib = b*. Then *bi = aiba = aba = i* so that for any open set *O*, *O = i(O) = bi(O) = b(O)* and hence the open sets of *(X, T)* are clopen.

Part (v) is trivial. If *(X, T)* is discrete then obviously *i = b*. If *i = b*, then *id = i = b* so that every subset of *X* is open in *T* and therefore *T* is the discrete topology [Gardner and Jackson]. □

From here, several more theorems similar in nature to Theorem 4 have been proven. Aull derived a plethora of results by restricting attention to Hausdorff spaces. A significant portion of literature has also been devoted to examining the preservation of the structure of Kuratowski monoids when considering subspaces or sums of spaces. It is not hard to see why the K-number of an open subspace of a topological space *(X, T)* is at most *K(X, T)*. However, if we do not confine ourselves to open subspaces then the problem becomes much more difficult. Many of these results can be found in Gardner and Jackson.

Furthermore, several variations of the Kuratowski Closure-Complement Theorem have been posed. The classical Kuratowski Closure-Complement Theorem, Theorem 1, considers only the closure and complement operators. A common theme in the literature has been to also consider the meet and join operators. In fact, for each subset *O* of $\{a, b, i, \wedge, \vee\}$, the problem of studying the monoid of a topological space produced by *O* is referred to as the Kuratowski *O*-problem. Again, results similar in nature to those presented here have been proven for these problems.

We conclude that many interesting properties of the closure and complement operators of a topological space can be derived by studying the Kuratowski monoid of the space. That these monoids are limited to only six possibilities, all of finite order, is quite surprising. As proven in Theorem 4, these six different monoids correspond to specific classes of topological spaces. Moreover, as mentioned previously, the Kuratowski Closure-Complement Problem is just one of an entire set of problems in topology referred to as the Kuratowski problems. Each of these problems deserves a discussion of its own and yields results just as surprising as those of the closure-complement problem.

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